

## Spectral condensation in laboratory two-dimensional turbulence

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(Received 2 September 2021; accepted 8 October 2021; published 19 October 2021)

Turbulence in two-dimensional flows is expected to produce a dynamical state in which energy condenses into the largest scale allowed by the system when the scale at which energy is dissipated exceeds the domain size. We study this phenomenon in a laboratory quasi-two-dimensional turbulent flow in a thin layer of electromagnetically driven fluid where the energy is primarily dissipated by bottom friction. By inserting boundaries of different sizes, we fix the driving and damping and vary only the domain size. Although we observe flow patterns that are consistent with previous claims of spectral condensation, we see no signatures in the energy spectrum. An analysis of the scale-to-scale energy flux reveals that small domains weaken the turbulent cascade, even though the bulk forcing and frictional dissipation remain the same. Our results suggest that we lack a robust set of criteria for the existence of spectral condensation, and that claims of condensation in experimental flows with small scale separations must be supported by strong evidence.

DOI: [10.1103/PhysRevFluids.6.104605](https://doi.org/10.1103/PhysRevFluids.6.104605)

### I. INTRODUCTION

Turbulent flows with a very high aspect ratio, where the vertical size of the flow domain is much smaller than the horizontal extent, are both of fundamental interest and relevant in applied settings. These kinds of high-aspect-ratio flows can often be considered to be two dimensional (2D) from a dynamical standpoint [1–3], particularly when geostrophic balance can also be invoked [4,5]. In particular, large-scale geophysical flows such as those in the atmosphere and oceans satisfy these conditions. It has been known for decades that 2D turbulence behaves very differently from its 3D counterpart [6]. While in 3D turbulence energy flows from the scale at which it is injected to smaller scales where it is dissipated by viscosity in the classical Richardson-Kolmogorov energy cascade [7], 2D turbulence displays the Kraichnan-Leith-Batchelor double cascade [6,8,9]. In this scenario, it is enstrophy that cascades to smaller scales and is dissipated by viscous effects; in contrast, energy cascades to larger length scales where it must be dissipated by some kind of large-scale damping mechanism [10,11]. However, if the scale at which this damping operates is larger than the largest scale of the system, there is no way to dissipate the energy and it is expected to condense into this lowest allowed mode in a phenomenon known as spectral condensation [6,11].

Spectral condensation has been investigated both in numerical simulations [12–14] and in experiments [15–18]. Even though energy condensation is only well defined in Fourier space (and we should be cautious about associating spectral concepts with physical-space signatures [19]), simulations conducted in domains with periodic boundary conditions have typically observed that the flow organizes into a large, coherent vortex dipole in the spectrally condensed regime [12–14]. Experimental studies, in contrast, have often reported the formation of vortex monopoles [15–18].

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It is not clear how to resolve this apparent discrepancy. It must also be noted that experiments always have nontrivial boundary conditions from the walls of the container and typically have spatial (and sometimes temporal) structure to their forcing. They can have a range of large-scale energy dissipation mechanisms that may not be well modeled by a simple linear friction [20–22]. They are also limited to fairly small scale separations [11]. Any of these confounding effects, which are not present in simulations, may make it difficult to tell whether a given flow is spectrally condensed or simply dominated by its boundaries or other experimental constraints that are not included in idealized theoretical or numerical descriptions.

Here, we revisit the question of 2D energy condensation using highly spatiotemporally resolved laboratory experiments in an electromagnetically driven thin-layer flow. By inserting barriers that restrict the domain size without changing the forcing or damping (which primarily arises from bottom friction), we independently vary the ratio of the length scale of frictional damping to the domain size while keeping the flow the same. We observe qualitatively different flow fields for domain sizes above and below the friction length, and the flow patterns in small domains are consistent with those that have been interpreted as indicators of spectral condensation. However, we find no associated signatures in the energy spectra. By examining the scale-to-scale energy flux using a filter-space technique, we find instead that the inverse energy cascade in our smaller domains appears to be weakened, and that limited scale separation and strong sidewall damping may prevent condensation. Our results indicate that one must be careful in interpreting limited results as signatures of spectral condensation in quasi-2D flow experiments, and that more robust indicators and clearer explication of the necessary conditions for true spectral condensation (in Kraichnan's original sense [6]) are needed.

## II. METHODOLOGY

### A. Experimental methods

We generate experimental quasi-two-dimensional turbulence using an electromagnetically driven thin-layer flow system and measure it with particle tracking velocimetry, as detailed elsewhere [2,23,24]. We drive flow in a thin layer of a solution of NaCl dissolved in water (14% by mass), with lateral dimensions of  $85 \times 85 \text{ cm}^2$  and a depth of 0.5 cm. We float a second layer of pure water on top of this electrolytic layer to remove surface tension at its upper boundary. The fluid sits on a glass floor coated with hydrophobic wax. A square lattice of permanent magnets with their poles oriented vertically lies below the glass floor. The magnets are spaced by  $L_m = 2.54 \text{ cm}$ . When we drive a dc electric current horizontally through the NaCl solution, a Lorentz body force is generated that produces fluid flow. Many of the previous experiments on condensation in similar systems arranged the magnet polarities in a checkerboard pattern so as to produce a vortex lattice at weak forcing [3,16,18]. Here, we instead arrange the magnets in stripes of alternating polarity that lead to alternating shear bands in a configuration sometimes known as Kolmogorov flow [25,26]. To quantify the strength of the forcing, we define a bulk Reynolds number as  $\text{Re} = u'L_m/\nu$ , where  $u'$  is the in-plane root-mean-square velocity and  $\nu$  is the kinematic viscosity.  $\text{Re}$  varies somewhat from experiment to experiment, but for most of the data reported here is around 200.

To measure the flow, we use particle tracking velocimetry (PTV), as we have outlined previously [26,27]. The NaCl solution is seeded with  $51\text{-}\mu\text{m}$ -diameter fluorescent polystyrene particles. Even in our strongest flows, the Stokes number of these tracers is on the order of  $10^{-4}$ , so that they are small enough to follow the flow accurately [22]. We excite the tracer fluorescence with blue LEDs, and image their motion with a 1-megapixel Point Grey Flea3 camera mounted above the apparatus. Images are captured at a rate of 60 frames per second, and we resolve roughly 45 particles per  $\text{cm}^2$ . Using these data, we can reconstruct spatiotemporally well-resolved Eulerian velocity fields. The mass density of the tracers is lower than that of the NaCl solution but higher than that of pure water, so the tracers remain trapped at the interface between the two fluid layers and move largely in a 2D plane. Nevertheless, some out-of-plane motion can occasionally occur. To remove its effects, we

project the measured flow fields onto a basis of numerically computed streamfunction eigenmodes to enforce the two-dimensionality of our data [2].

To study the effects of domain size on spectral condensation, we added additional square boundaries to the center of the apparatus. These boundaries were designed to allow the passage of electric current but not fluid, so that they change the flow domain without changing the forcing. We used insertable boundaries with edge sizes ranging from 5 cm (roughly two magnet spacings  $L_m$ ) to 20 cm.

### B. Filtering and energy flux

Spectral energy condensation is a dynamical process that requires energy to move between the various scales of turbulent motion via the energy cascade. Thus, static characterizations of the apportioning of energy between scales such as spectra may not always convey sufficient information, particularly in experiments where the scaling range is typically small [11].

An alternative approach that directly gives access to the energy flux between scales is to use a so-called filter-space technique (FST) [28–35]. The principle underlying an FST is straightforward. As is well known from large-eddy simulation (LES), applying a low-pass filter to the Navier-Stokes equations introduces a turbulent stress term that accounts for momentum transfer between the retained and removed scales of the flow [7]. In an FST, unlike in an LES, a low-pass filter is applied to measured (or simulated) flow fields only as an *a posteriori* step, allowing the measurement of this turbulent stress in a spatiotemporally resolved way and at any desired scale.

We implement an FST by convolving our measured Eulerian velocity fields with a kernel  $G^{(r)}$  that acts as a low-pass filter in Fourier space, suppressing rapid spatial variation while retaining slow spatial variation. The superscript  $r$  denotes the length scale below which variation is suppressed. Our results are not qualitatively sensitive to the exact form of the kernel [31,36]; here, we use an isotropic finite impulse response filter constructed from a sharp spectral filter with a cutoff wavenumber of  $2\pi/r$  smoothed with a Gaussian window function to avoid ringing. With these definitions, a single component of the filtered velocity field is given by

$$u_i^{(r)}(\mathbf{x}) = \int G^{(r)}(\mathbf{x} - \mathbf{x}') u_i(\mathbf{x}') d\mathbf{x}', \quad (1)$$

where again the superscript  $r$  indicates the scale at which the field has been filtered. We define the filtered kinetic energy as  $E^{(r)} = \frac{1}{2} u_i^{(r)} u_i^{(r)}$ , and by inspecting its equation of motion [7,30,31,34] define the scale-to-scale energy flux as

$$\Pi^{(r)} = -[(u_i u_j)^{(r)} - u_i^{(r)} u_j^{(r)}] \frac{\partial u_i^{(r)}}{\partial x_j}, \quad (2)$$

where the term in square brackets is the turbulent stress tensor  $\tau_{ij}^{(r)}$ . With our sign conventions,  $\Pi^{(r)} < 0$  denotes a flux of energy from small to large scales (inverse cascade), while  $\Pi^{(r)} > 0$  indicates that energy is transferred from large to small scales (forward cascade).

## III. RESULTS

Previous studies of spectral energy condensation in 2D turbulence, and particularly previous experimental studies, have often argued that condensation is occurring based on the flow patterns that are observed. As mentioned above, numerical simulations in periodic domains often display vortex dipoles [12–14] while experiments have instead found vortex monopoles [15–18]. Thus, before discussing the spectral properties of our flows, we show example flow fields to give a qualitative picture of the flow as we change the domain size.

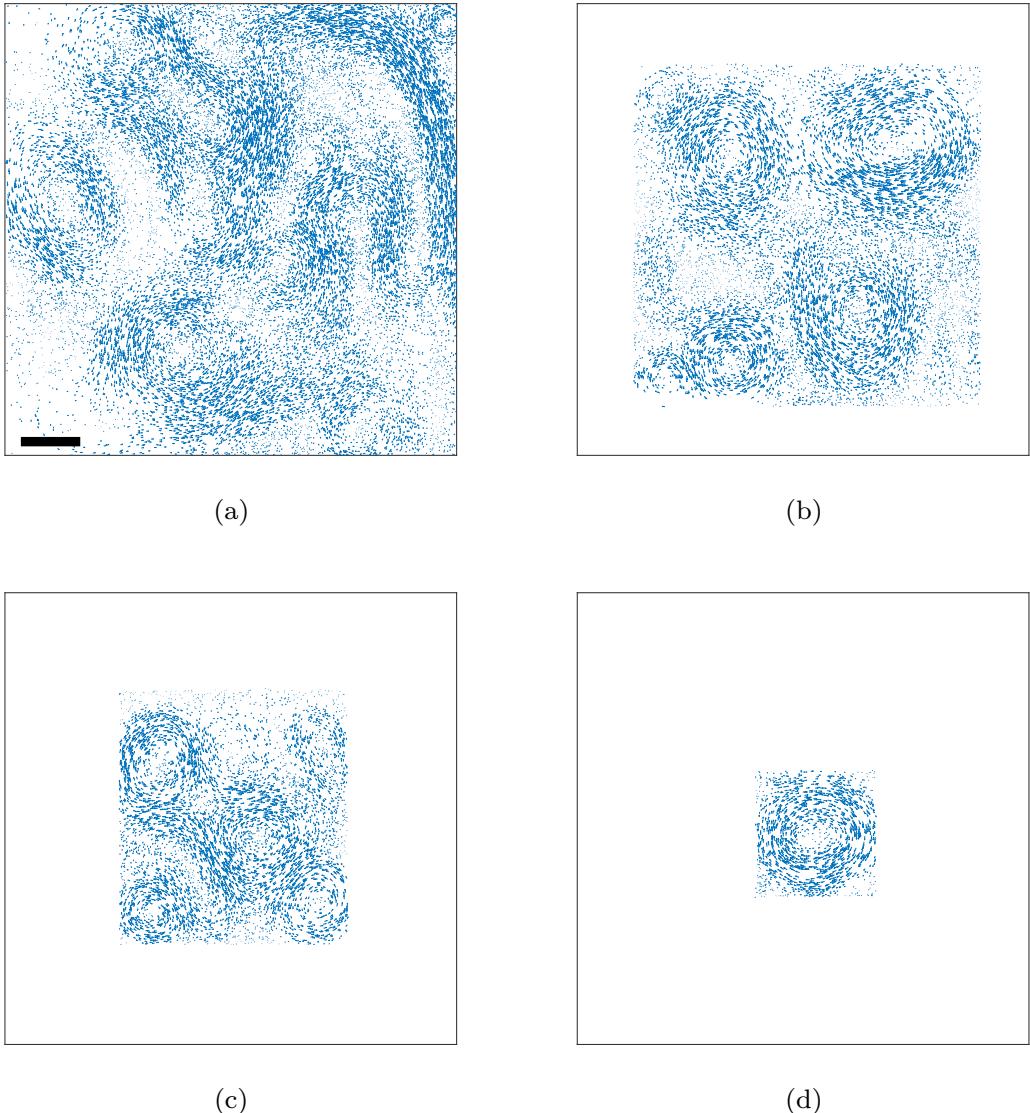


FIG. 1. Representative instantaneous velocity fields observed for four domain sizes: (a)  $20 \times 20 \text{ cm}^2$ , (b)  $15 \times 15 \text{ cm}^2$ , (c)  $10 \times 10 \text{ cm}^2$ , and (d)  $5 \times 5 \text{ cm}^2$ . The scale is the same for all four panels, and the scale bar shows the magnet length scale  $L_m = 2.54 \text{ cm}$ .

In Fig. 1, we plot instantaneous flow fields for each of the four domain sizes we considered. Qualitatively, the velocity field in the largest domain [Fig. 1(a)] is different from the others. Unlike the fields in the smaller domains in which regular patterns of vortices are clearly visible, the velocity field in the largest domain is disordered with no clear vortex patterns. Recall that the parameters of our experiment (most relevantly the fluid layer depth and the forcing strength and scale) are identical in these four cases; all that changes is the size of the flow domain.

Unlike in 3D turbulence, where the length scale of dissipation adjusts dynamically to the flow conditions, in our laboratory realization of 2D turbulence, where the dominant energy dissipation mechanism is bottom friction, the length scales of both the forcing and the energy dissipation are fixed by the apparatus. The length scale at which energy is injected is close to  $L_m$ , the spacing of

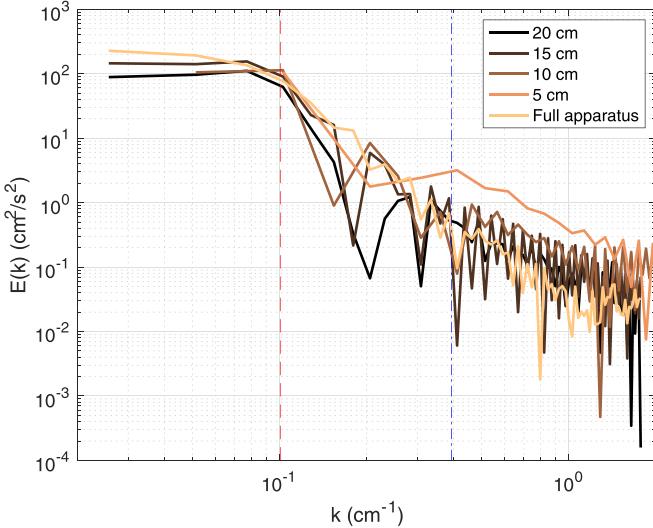


FIG. 2. Energy spectra  $E(k)$  as a function of wave number  $k$  for flows in our four restricted domains as well as in the full apparatus with no inserted boundary. The vertical dotted-dashed blue line indicates the injection wave number, and the vertical dashed red line indicates the frictional energy dissipation wave number.

the magnets [37].  $L_m$  is smaller than the domain for each case in Fig. 1. The length scale at which friction acts can be written as [11,18]

$$L_\alpha = \epsilon^{1/2} \alpha^{-3/2}, \quad (3)$$

where  $\epsilon$  is the rate of scale-to-scale energy transfer in the inverse cascade range and  $\alpha$  is the linear dissipation rate due primarily to bottom friction. To estimate  $\epsilon$ , we use an FST [38]. To estimate  $\alpha$ , we measure the decay rate of the flow in our apparatus when the forcing is switched off, as we have described previously [22]. Combining these measurements, we find  $L_\alpha = 16.4$  cm—smaller than the domain size in Fig. 1(a), but larger than the domain sizes in Figs. 1(b)–1(d). Thus, we can surmise that spectral condensation is possible in our three smaller flow domains, and that this may be the reason why the flow fields look different. However, purely qualitative observations of the flow fields are not strong evidence of condensation.

To look for more quantitative signatures of energy condensation, we follow previous studies [3,11,14] and turn to the energy spectrum. Spectra for the four flows with restricted domains illustrated in Fig. 1 as well as for flow in the full apparatus are shown in Fig. 2. Perhaps surprisingly given the flow fields shown in Fig. 1, the spectra for the different domain sizes are not readily distinguishable. In particular, for none of the cases do we see a sharp increase in the energy content at low wave number, in stark contrast to what has been reported in other experiments [3,18]. However, this result should not necessarily be unexpected given the small separation of scales in our experiment (and indeed in all laboratory realizations of 2D turbulence). As we have discussed, in most laboratory realizations of 2D turbulence both the injection and dissipation scales are set by experimental constraints, and are not dynamical. In our case, the ratio  $L_\alpha/L_m \approx 6.5$ , far less than what would be required to see robust scaling. With such a small potential scaling range, spectral features are likely to be strongly influenced by the exact method used to compute the spectra, and may be serendipitous. For the cases where we might expect to see spectral condensation, the ratio between the smallest allowed wave number and the injection scale is even smaller. Thus, it is much more likely that the coherent vortical flows we observe in Fig. 1 in the nominally condensed cases are a result of strong contamination from the sidewall boundaries and not spectral condensation. Note that these considerations do not necessarily imply that our flows in the restricted domains are

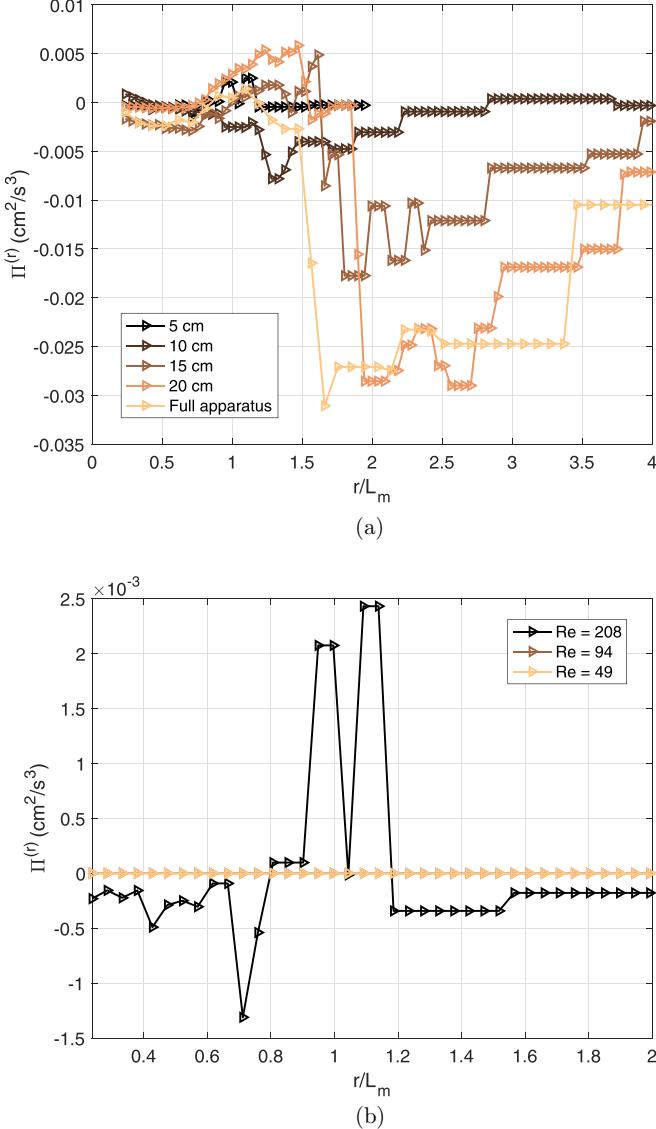


FIG. 3. Spatiotemporally averaged spectral energy fluxes  $\Pi^{(r)}$  as a function of filter scale  $r$ , normalized by the magnet spacing  $L_m$ . (a) Spectral energy fluxes for different domain sizes, including the full apparatus with no inserted boundaries. Negative values indicate transfer to larger length scales. (b) Spectral energy flux for the  $5 \times 5 \text{ cm}^2$  domain with the same fluid depth but different Reynolds numbers. The flow fields for each Reynolds number display vortex monopoles, but the spectral energy flux is markedly different.

*not* spectrally condensed; rather, they suggest that neither the flow structure nor the spectrum are appropriate metrics for assessing spectral condensation given typical experimental constraints.

Measurements of the spectral energy flux using FSTs tell a more nuanced story. In Fig. 3(a), we show the spatiotemporally averaged spectral energy flux  $\Pi^{(r)}$  as a function of filter scale  $r$  for the cases whose spectra are shown in Fig. 2. All cases aside from the  $5 \times 5 \text{ cm}^2$  domain show at least some net inverse energy flux for scales larger than the injection scale, as we have found previously in this apparatus [22–24,32,37]. The energy flux for the two cases with domains larger than the

friction scale  $L_\alpha$  (namely the full apparatus and the  $20 \times 20 \text{ cm}^2$  domain) is similar, suggesting as expected that flows in domains larger than the scale of frictional dissipation are fairly similar because any scales larger than  $L_\alpha$  do not affect the turbulent cascade. As the domain size shrinks below  $L_\alpha$ , however, the energy flux changes even though the spectra remain largely similar. In particular, the magnitude of the energy flux monotonically decreases as the domain size shrinks, suggesting that the presence of the sidewalls is weakening the inverse energy cascade. Thus, for our flow configuration, the effect of making the flow domain smaller than the length scale of frictional dissipation appears not to be a pileup of energy in the lowest wave number but rather a weakening of the turbulent cascade.

Although exactly why this occurs is difficult to ascertain, we offer two potential explanations for this observation. First, we note again that there is simply not much scale separation in experimental realizations of 2D turbulence. In the best possible scenario for observing spectral condensation, the dissipation scale would be on the order of the full apparatus size. In our case, however, even that would only give a ratio of the friction scale to the injection scale of about 30—and our apparatus is relatively large compared to other quasi-two-dimensional turbulence experiments. In the actual case in our experiment with a scale ratio of about 6.5, the inverse energy cascade is never fully developed, as the range of scales is simply not large enough. Decreasing the possible size of the inverse energy cascade range further by making the domain smaller may simply destroy the weak turbulence rather than allow the dynamical spectral energy condensation mechanism to operate. A second, related possible explanation for our results is that the introduction of fixed lateral sidewalls close to the domain where we observe the turbulence may introduce an additional frictional dissipation mechanism operating on a different scale. The frictional dissipation rate  $\alpha$  characterizes dissipation by the interaction of the flow with the no-slip bottom surface of the apparatus; but when the no-slip sidewalls are close to the bulk of the flow, they may also play a role in frictional dissipation. It is reasonable to expect that their influence will grow and potentially dominate the bottom friction when the domain is small. Suggestively, if we take the expression for the friction scale given in Eq. (3) and evaluate it not with  $\epsilon$  as measured for the flow with no inserted boundaries but rather as measured for the domains smaller than the bulk estimate of  $L_\alpha$ , we find friction scales nearly equal to the linear size of the domain, perhaps indicating the strong influence of the sidewalls for these small domains. If this explanation holds, it would again argue that the coherent flows we observe in Fig. 1 for our smaller domains are largely due to boundary conditions rather than a dynamical mechanism such as spectral condensation.

To provide a final piece of cautionary evidence against overinterpreting the appearance of coherent flow patterns in 2D flow, in Fig. 3(b) we plot the spectral energy flux as a function of filter scale for our smallest domain (the  $5 \times 5 \text{ cm}^2$  case) for three different bulk Reynolds numbers. In all cases, the flow fields look qualitatively similar; for reference, the flow field for the largest Reynolds number is shown in Fig. 1(d). However, only this largest Reynolds number shown any amount of scale-to-scale energy transfer, and even that is small [as compared with larger domains, as shown in Fig. 3(a)]. One would certainly not expect vigorous turbulence and its associated dynamics in a flow that is only about twice as large as the injection length scale; but, in the context of 2D turbulence, the flow pattern is similar to what has been reported as a signature of spectral condensation.

#### IV. SUMMARY AND CONCLUSIONS

By varying only the domain size while keeping the driving and damping fixed, we have investigated the possibility of spectral condensation in our quasi-two-dimensional laboratory flow. Although the flow patterns observed in cases where one might expect spectral condensation based on an estimate of the friction length scale are consistent with what have been interpreted as indicators of spectral condensation in past experiments, we find no associated signature in the energy spectra. Instead, by examining the scale-to-scale energy flux, we find that the primary effect of limiting the domain size appears to be a weakening of the inverse energy cascade, which is not fully developed even in our full apparatus.

Our results suggest that one must be cautious in interpreting features such as large-scale coherent flows or spectra observed in laboratory realizations of 2D turbulence through the lens of Kraichnan's classical picture of spectral energy condensation. Laboratory flows are always limited relative to asymptotic theory or numerical simulations in the scale separation that can be achieved, so that the inverse energy cascade that is the underlying mechanism driving spectral condensation is not fully developed. Laboratory flows also have boundaries and nontrivial forcing, which may impose constraints on the large-scale flow patterns. Finally, and perhaps most importantly, laboratory flows may contain a variety of different large-scale energy dissipation mechanisms that are not represented in theory or simulations and that may fundamentally limit comparisons between current experiments and classical theory.

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